IIT Research Institute

Technical Note No. CSTN-106

PIECE-WISE LINEAR APPROXIMATIONS

by

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ABSTRACT

A computational algorithm for the determination of a piece-wise linear approximation to an arbitrarily specified function of one variable is described. In particular the algorithm generates the optimal piece-wise linear approximation consistent with a specified accuracy in the sense that the number of segments is minimized. It is demonstrated that in contrast to the problem of minimizing the maximum error with a specified number of segments, this formulation leads to a computation based only on local values of the given function and a correspondingly efficient computational procedure.

Piece-Wise Linear Approximations

The representation of arbitrarily given curves in terms of piece-wise linear approximations is a technique of importance not only in many digital computer applications, but is widely employed in analog computer embodiments for function representation. For many applications in the digital realm it is a strong competitor to rational function approximation techniques both with respect to the compactness of the representation and the speed with which the approximation may be evaluated. Of particular interest on both counts, speed and compactness, are approximations comprised of the minimum number of linear segments consistent with the level of accuracy desired. While the problem of discovering a minimum segment approximation to an arbitrarily specified function has been previously treated by mathematical programming techniques (see for example Gluss [1] where it is formulated as a dynamic program), the procedure described herein treats the problem rather differently and leads to an efficient computational algorithm.

The particular problem which is considered in this paper is that of approximating a single valued function of one variable in terms of a sequence of linear segments. More formally we assume a given function $f(x)$ defined on the interval $[a,b]$, and we seek an approximating function $g(x)$ such that $|f(x)-g(x)|\leq$ for $a\leq x\leq b$. The function $g(x)$ is to consist of a sequence of linear pieces so that for every x in $[a,b]$, $g(x)$ is of the form $g(x)=\alpha+\beta x$. More particularly values x_i for $i=0,1,2,\ldots,n$ are to be selected having the properties that $a=x_0$,

$x_i < x_{i+1}$, and $x_n = b$, so that for $x_{i-1} \leq x < x_i$, $g(x) = g_i(x) = \alpha_i + \beta_i x$. Of all approximating functions g(x) which meet the above conditions we are interested in selecting one which is comprised of the minimum possible number of linear segments. Two cases are to be considered; (1) the function g(x) may be discontinuous at the boundaries between the linear segments, (2) the function g(x) must be continuous everywhere on [a,b].

Consider first the case in which discontinuities are permitted at the linear segment boundaries. An example of a three segment approximation of this variety is given in Figure 1. In this case the algorithm may begin by selecting the longest single linear segment which satisfies the error restriction everywhere from a to the right-most boundary of the initial segment, i.e., by finding α_1, β_1, and x_1 such that $|(\alpha_1 + \beta_1 x) - f(x)| < \delta$ for $a \leq x \leq x_1$ and moreover x_1 is as large as possible. It is clear that the segment $g_1(x) = (\alpha_1 + \beta_1 x)$ for $a \leq x \leq x_1$ may be taken as a component of the optimal approximation since any other choice of a left-most segment would result in a "remaining function to be approximated" which includes the interval $[x_1, b]$. It is obviously not possible that the number of linear segments required to approximate f(x) over $[\xi, b]$ is less than the number required over $[x_1, b]$, where $\xi < x_1$, since any M segment approximation of f(x) over $[\xi, b]$ is itself an approximation over the subinterval $[x_1, b]$ consisting of at most M segments.

The algorithm proceeds by repeated selection of the linear segment of maximum scope from the left-hand end of the "remaining function to be approximated".

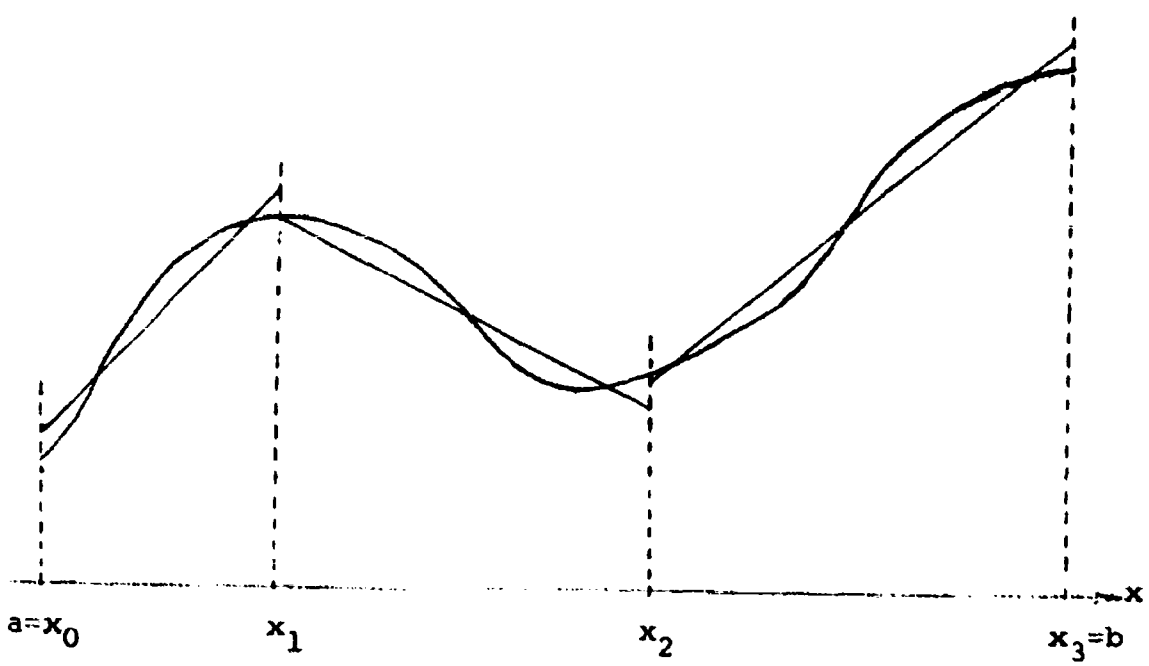

Fig. 1 A three segment approximation with boundary discontinuities

Since we are primarily concerned with the determination of approximations to arbitrarily given functions by means of a digital computer, we will deal with a slightly different problem in which we assume that the function f(x) is defined only at a discrete set of points which we take to be the non-negative integers. Thus we restate the problem of selecting the initial linear segment as follows. Given real numbers δ,f(0),f(1), f(2),..., find real numbers α, β, and n such that $|f(j)-(\alpha + \beta j)| \leq \delta$ for $0 \leq j \leq n$ and n is maximal.

Note that for each value of j in $0 \leq j \leq n$ it follows that each of the following relations must hold:

$$\alpha + \beta j \leq f(j) + \delta, \text{ and}$$

$$\alpha + \beta j \geq f(j) - \delta.$$

Note that each of the above relations may be regarded as defining a half-space in the two dimensional space whose coordinate axis are α and β. Thus each value of j may be associated with two half-spaces the intersection of which defines a strip in the α,β plane. This strip includes the parameters of all of the linear segments which meet the error conditions for that particular value of j. Clearly values of α and β exist satisfying $|f(j) - (\alpha+\beta j)| \leq \delta$ for all $0 \leq j \leq n$, just in case the intersection of all the associated half spaces is not empty. This observation suggests the computational algorithm. We begin with j=0 and determine the equations of the corresponding half planes. We next set j=1 and find the "corners" of the convex polygon which is the intersection of the 4 half planes. Note that at this

point the convex polygon will be in fact a parallelogram. We next increment j by one and determine the resulting cumulative intersection which will be either null or a convex polygon. We proceed in this manner until we have found the largest value of j such that the cumulative intersection is not empty. Having found the largest such value of j we may take as the parameters of the initial linear segment any value of α and β within the corresponding polygon.

While the preceding discussion outlines an algorithm which is of interest in the determination of piece-wise linear approximations in the case where discontinuities are permitted, a relatively minor variant of that procedure handles the more interesting case in which g(x) is continuous. While the principle reason for our interest in the continuous case is that discontinuities frequently are unacceptable because of the way in which the approximation is employed in subsequent computations, the continuous approximation has the interesting side benefit that it leads to a more compact representation. This is due to the fact that although the number of linear segments in a continuous approximation may be somewhat greater than in the discontinuous case each segment may be defined by two rather than three parameters. To illustrate the procedure for a continuous approximation we return to the consideration of a continuous f(x) and investigate some of the properties of the linear segment such that $|(\alpha + \beta x) - f(x)| \leq \delta$ for $a \leq x \leq x_1$ for the largest possible x_1. First, it is clear that $|f(x_1)-(\alpha+\beta x_1)| = \delta$ since, if this were not the case, it would be possible to

to increase x_1. Let us assume that $\alpha+\beta x_1=f(x_1)+\delta$. A theorem due to Tchebysheff tells us that there must be at least 3 points at which the error takes on its maximum value and that the sign of the error at these points alternates as we move from one point to the next. Thus there is a right-most point $x_0<x_1$ at which $\alpha+\beta x_0=f(x_0)-\delta$. We show in Appendix C that the point x_0 has the interesting property that there is in fact no point $z>x_0$ such that $\alpha+\beta z=f(z)-\delta$ and for all $a\leq x\leq z$, $|f(x)-(\alpha+\beta x)|\leq\delta$.

Suppose that by some process we have determined the parameters of the left-most line segment, that it remains within the error bound for $a\leq x\leq y_1$, and at y_1 we have $\alpha+\beta y_1=f(y_1)+\delta$. Suppose further that y_0 is the right-most point in the interval (a, y_1) at which the linear segment touches the lower error boundary, i.e., $\alpha+\beta y_0=f(y_0)-\delta$. We now observe that the remaining problem is that of finding a set of connected line segments, minimum in number, such that for $y_0\leq x\leq b$, $g(x)\geq f(x)-\delta$, and for $y_1\leq x\leq b$, $g(x)\leq f(x)+\delta$. Any set of connected line segments which meets the above condition will contain a line which intersects the line $(\alpha+\beta x)$ along the segment joining the point $(y_0, f(y_0)-\delta)$ to the point $(y_1, f(y_1)+\delta)$. Thus if we define this point of intersection as the right-most boundary of the initial segment we are assured of continuity at the boundary and over the segment. We next note that the difficulty (measured in terms of the number of segments required) of the remaining approximation problem (after the first segment has been selected) can only be reduced by moving either or both of the points y_0 and y_1 to the right, since this diminishes the range over which

the approximation is constrained by the upper and/or lower error boundaries. Since the result of Appendix C shows that both y_0 and y_1 are maximized by the same linear segment we cannot do better than to use that segment.

We note that after the first linear segment has been established the remaining approximation problem differs from the original problem in that the upper and lower error boundaries do not begin at the same point. This does not complicate the problem in any essential way, however, and may be easily taken into account by selecting the appropriate half-spaces to enter into the computational algorithm.

An intuitively satisfying characterization of the procedure is illustrated in Figure 2 in which the upper and lower boundaries are regarded as defining a two dimension tube into which a straight stick is pushed from the left. Figure 2a illustrates the terminal position of the stick. This position of the stick defines the first line in the approximating sequence. That part of the upper boundary to the left of point b and that to the left of point a on the lower boundary are then cut away and the process is repeated. This leads to the terminal stick position of Figure 2b which defines the second line in the sequence. The articulation point between the first two sections is obviously their point of intersection. A computational algorithm treating the discrete case is described in Appendix A. An efficient procedure for handling the intersection of a half-space and a convex polygon

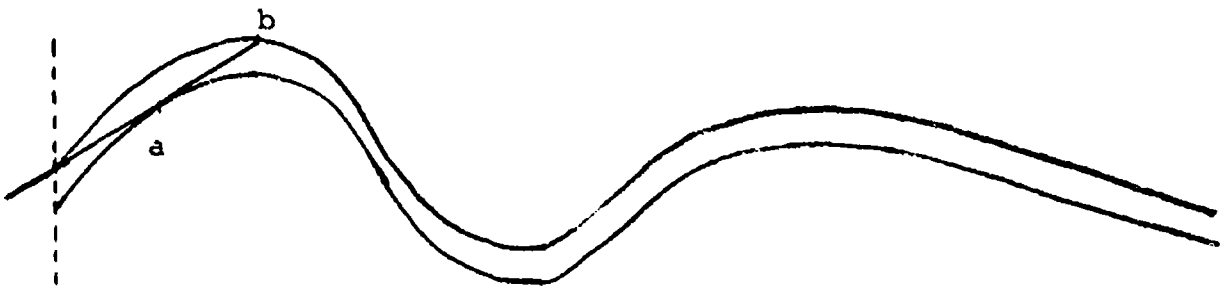

Fig. 2a Initial Segment Determination

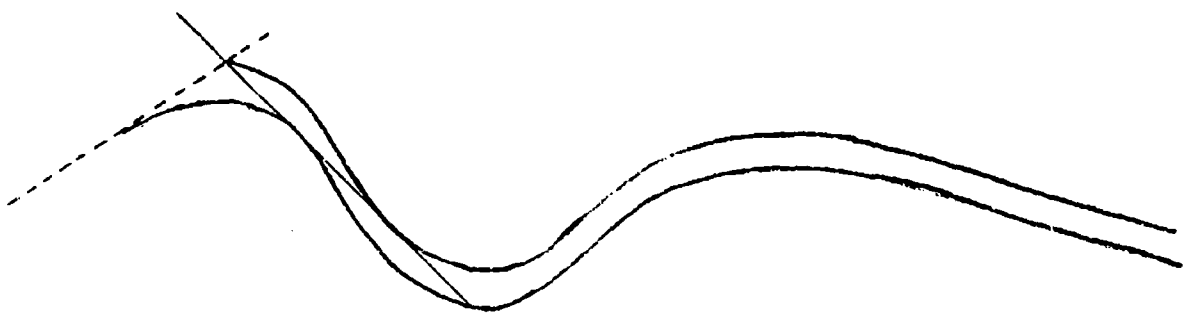

Fig. 2b Second Segment Determination

is given in Appendix B, while Appendix C presents the theorem on the basis of which the optimality proof rests.

While the preceding discussion has been in terms of a uniform band of tolerable error it should be apparent that the uniformity of the error is in no way essential to the method and in fact the computational algorithm outlined in Appendix A is given in terms of an arbitrary "upper" and "lower" curves. While these curves may be defined as $(f(x)+\delta)$ and $(f(x)-\delta)$ respectively they may also be established by an error condition which depends on either the independent or dependent variable. Thus there may be regions of x in which the error is not particularly important or the tolerable error may be more reasonable specified as a given percentage of the independent variable.

It is also possible to employ a variant of this technique to represent general paths in 2 or more dimensions where the path is specified by the coordinates of an ordered set of points along the curve. This case may be handled by using the algorithm of Appendix A to obtain approximations to the parametric functions which describe the variation of each of the coordinates as functions of the index variable over the original point set. The index variable may then be eliminated between the parametric approximations and a piece-wise linear approximation to the general path in 2 or more dimensions may be obtained.

(1) B. Gluss, "A Line-segment curve-fitting Algorithm" Information and Control, Vol. 5, No. 3, Sept. 1962

Appendix A

An Algorithm for the Determination of Piece-Wise Linear Approximations

The problem with which this algorithm deals is illustrated in Fig. 3. C1(M) is ordered set of points defining the "bottom" curve and C2(M) is an ordered set of points defining the top curve. M is employed as an index variable over both C1 and C2. What is required is a set of lines which define a chain of connected line segments such that all of the points on C1 are "under" the chain and all C2 points are "over" the chain. The chain is defined by a sequence of lines LL(I),I=1,2,... where the first segment is that part of LL(1) between its intersection with the vertical line through M=0, and its intersection with LL(2), the next segment in the chain is that part of LL(2) between its intersections with LL(1) and LL(3), etc.

The computational algorithm is given in flow diagram form in Fig. 4. With respect to Fig. 4 the following definitions are in order.

Each of the lines in Fig. 3 may be defined in terms of a y intercept and a slope, e.g. LL(1) may be defined by the relation $y=\alpha_1+\beta_1 M$. Thus we may define a two dimensional space with axes α and β, each point of which corresponds to a line in Fig. 3. The notation POLY X is intended to refer to the

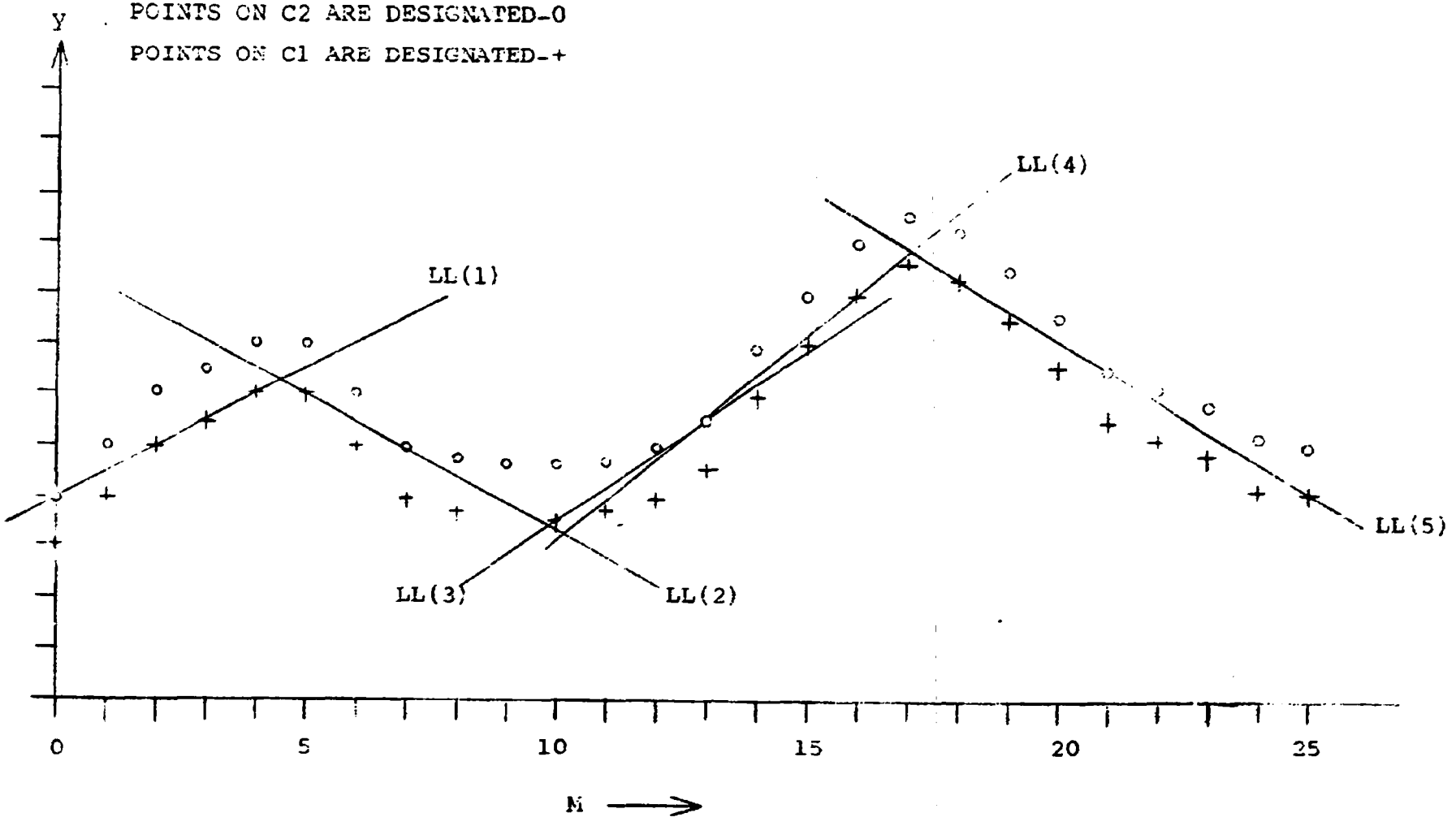

FIGURE 3 - THE DISCRETE CASE

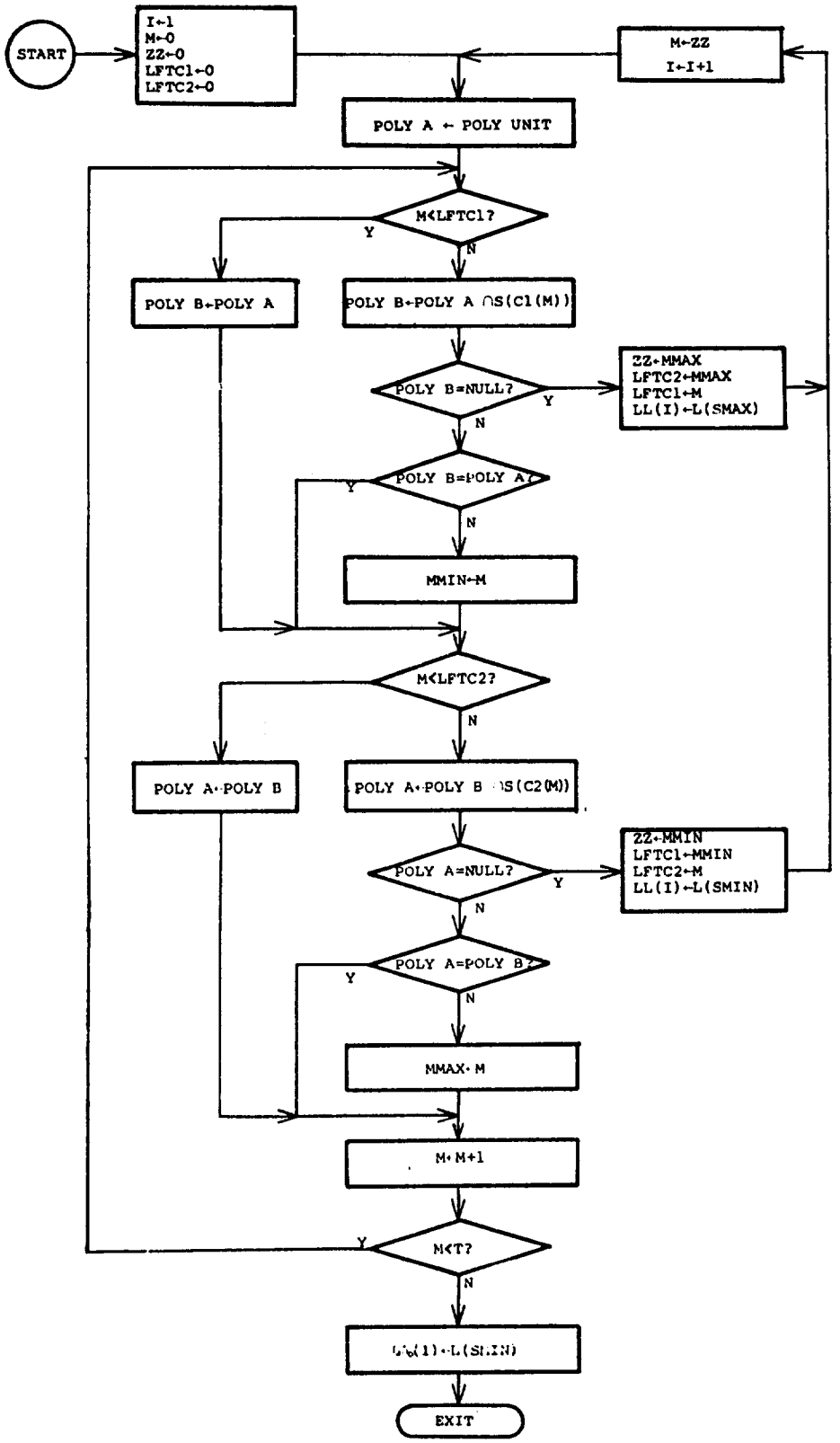

FIGURE 4 - FLOW DIAGRAM FOR APPROXIMATION ALGORITHM

interior and the boundary of a convex polygon in α,β space. POLY INIT is intended to denote a very large, essentially infinite, convex polygon which contains in its interior the points which correspond to any line which could conceivably be a member of the approximating sequence. The requirement that $\alpha_1 + \beta_1 M \leq C2(M)$ for each particular value of M may be regarded as defining a half-space in the α,β plane, i.e. the half-space in which the relation holds, and this half-space is denoted by S(C2(M)). The substitution operation POLY A←POLY B ∩S(C2(M)) means that POLY A is replaced with the polygon which results from the formal intersection of POLY B and the half space S(C2(M)). The variable LFTC1 is employed to denote the index of the left-most point in the set C1 which is to be considered as a constraint on the line segment being determined. LFTC2 is similarly employed to denote the left-most point on C2. Thus with respect to Fig. 3 for LL(1), LFTC1=LFTC2=0 since the first line must be constrained by the points C1(0) and C2(0) and all points to the right insofar as possible. The general strategy is to index M and to determine at each step the polygon in the α,β space which bounds the region in which acceptable line parameters are to be found. Thus in Fig. 3 the intersection of the 12 half-spaces, given by $\alpha_1+\beta_1 M \geq C1(M)$ and $\alpha_1+\beta_1 M \leq C2(M)$ for M=0,1,2,3,4,5 are found to define a non-null convex polygon while the intersection of that polygon with the half-space defined by $\alpha_1+\beta_1 .6 \leq C2(6)$ is null. This allows us to choose

parameters for LL(1). In particular since LL(1) fails to meet the constraint for a point on the upper curve, we choose for the parameters of LL(1) the least slope feasible point in the polygon which defines the feasible region for all points from M=0 through M=5, i.e. the line LL(1) will come as close as possible to satisfying C2(6). MMIN is defined to be the index of the point on C1 which determines the minimum slope. In the case of LL(1) in Fig. 3, MMIN=3. Similarly MMAX is the index of the point on C2 which determines the maximum allowable slope. SMAX is intended to denote that point on the boundary of a polygon in the α,β plane for which the slope is maximum while SMIN is similarly intended as the point of minimum slope. The notation L(SMAX) or L(SMIN) refers to the line in the y,M plane which corresponds to the point SMAX in the α,β plane. Thus in the case of the first line in Fig. 1 LL(1) is choosen as L(SMIN) for the appropriate polygon. The variable ZZ is reevaluated for each approximating line and is the smallest value of M which need be considered in the determination of that line, i.e. ZZ=MIN(LFTC1,CFTC2). For example for the determination of LL(2) in Fig. 3, LFTC1=3, LFTC2=6, and ZZ=3. Thus LL(2) must be above every point on C1 from M=3 on to the right as far as possible while LL(2) need be below points on C2 from M=6 on to the right. The variable T denotes the total number of points for which C1 and C2 are defined. The condition M=T terminates the algorithm.

Appendix B

An Algorithm for Calculating the Intersection of a Convex Polygon and a Linear Half-Space

The intersection of a linear half-space and a convex polygon is either null or a convex polygon. The input data on which the algorithm is based is a description of a convex polygon plus a description of the half space, the output is either a description of a convex polygon or an indication that the intersection is null. Since the algorithm is to be used recursively we require that the format of the output description be the same as that of the input description. The format for the specification of the convex polygon is an ordered list of the "corners" of the polygon (with the first corner repeated as the last corner) plus an indication of the number of such corners. The notation employed for the input polygon in the flow diagram of Fig. 5 is n for the number of corners, and A_0, A_1, ..., A_n for the coordinates of the corners with $A_0=A_n$. The notation for the resultant polygon is B_0, B_1, ..., B_m, with $B_0=B_m$. The half-space, S, is denoted by the equation of the separating line L. The term ($A_i \in S$?) asks whether the point A_i is an element of the half space defined by the boundary L. The term LN(A_i, A_i-1) denotes the line segment joining the point A_i to the point A_i-1 and the term

(L LN(A_i, A_i-1)) denotes the point of intersection of the separation line L and the segment L (A_i, A_i-1). A geometric illustration of the input and output polygons is given in Fig. 6. The algorithm itself is illustrated in the flow diagram of Fig. 5. The algorithm is based on the fact that the "corners" of the resultant polygon will be the corners of the input polygon that are in the half-space plus the points of intersection of the separating boundary and the boundaries of the input polygon.

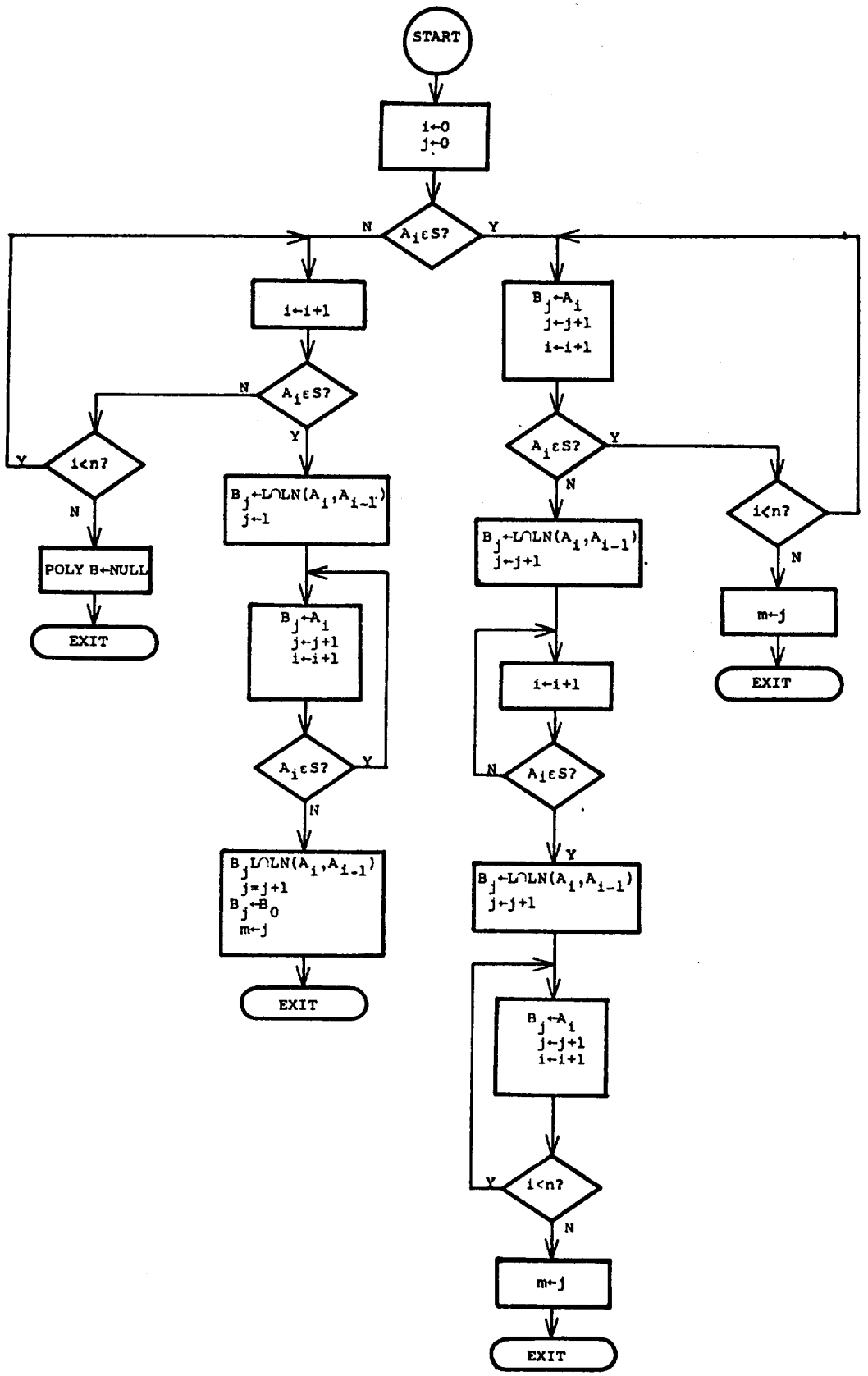

FIGURE 5 - FLOW DIAGRAM OF POLYGON INTERSECTION ALGORITHM

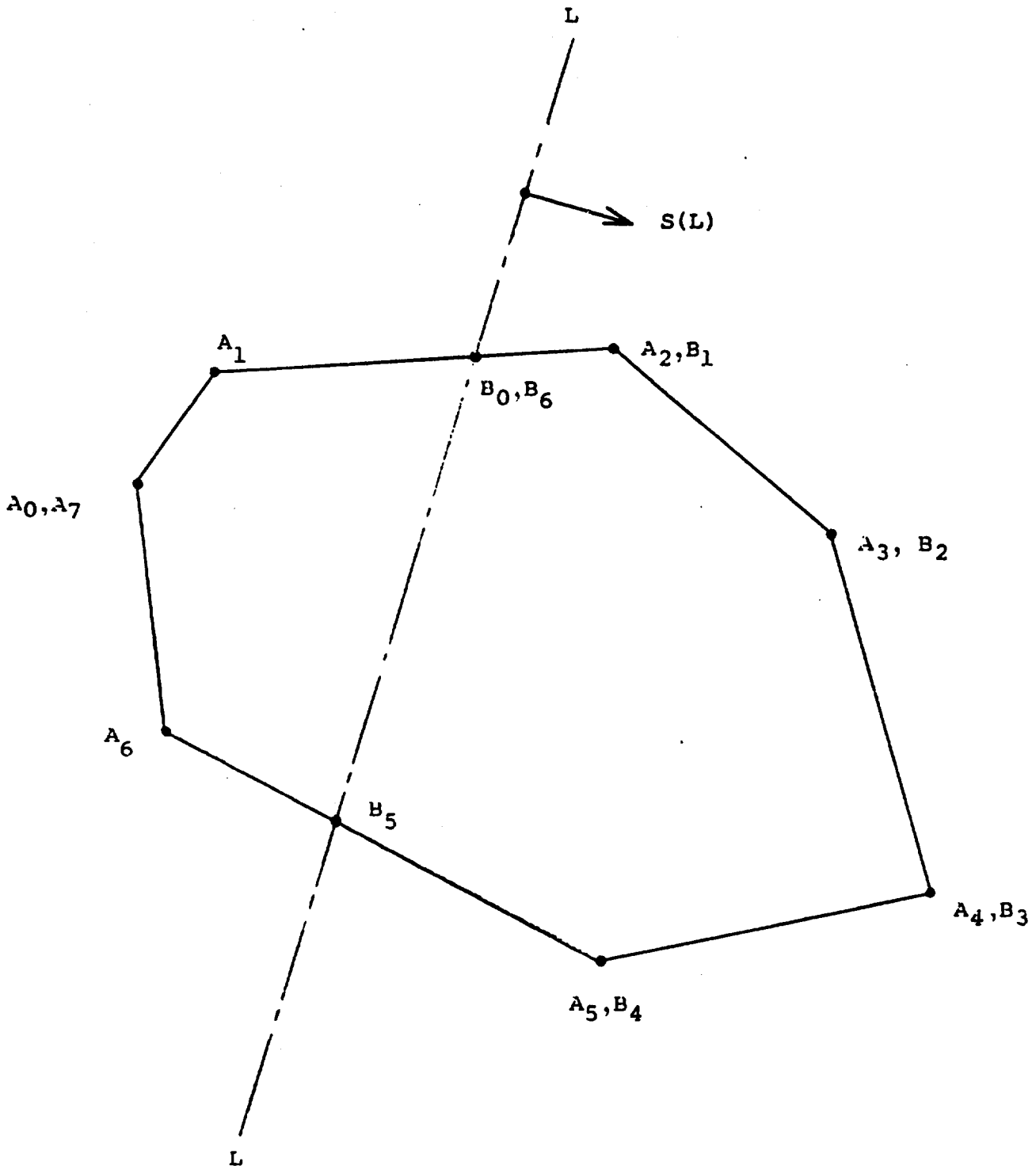

FIGURE 6 - GEOMETRIC INTERPRETATION OF INTERSECTION ALGORITHM

Appendix C

A Theorem on Tchebysheff Approximation

Let $f(x)$ be a continuous real valued function defined everywhere on the non-negative axis. Let $g(x)$ be an arbitrary polynomial of degree n and let G be the set of all such nth degree polynomials. Define $R(f,g)$ to be the largest value of x such that for all $0 \leq y \leq x$, $|f(y)-g(y)| \leq \delta$. If $|f(0)-g(0)| > \delta$, then $R(f,g)$ is defined to be zero. Let x_R be defined as

$$x_R = \max_{g \in G} R(f,g),$$

and denote by g_R the polynomial which maximizes R, i.e.,

$$x_R = R(f,g_R).$$

Define $T(f,g)$ to be the maximum value of x such that for all $0 \leq y \leq x$, $|f(y)-g(y)| \leq \delta$ and further $g(x)=f(x)+\delta$. Let x_T be defined as

$$x_T = \max_{g \in G} T(f,g),$$

and denote by g_T the polynomial which maximizes T, i.e.,

$$x_T = T(f,g_T).$$

Define $B(f,g)$ to be the maximum value of x such that for all $0 \leq y \leq x$, $|f(y)-g(y)| \leq \delta$, and further $g(x)=f(x)-\delta$. Let x_B be defined as

$$x_B = \max_{g \in G} B(f,g)$$

and denote by g_B the polynomial which maximizes B, i.e.,

$$x_B = B(f, g_B).$$

Less formally, if we think of an upper boundary given by $f+\delta$ and a lower boundary given by $f-\delta$, g_R is the polynomial which stays within the error band, i.e., the interval between $f+\delta$ and $f-\delta$, as far to the right as possible and x_R the point at which it leaves the error band. g_T is the polynomial which stays in the error band everywhere from zero to x_T at which point it contacts the "top" boundary. x_T is as far to the right as possible. g_B is similarly defined except that at the point x_B the polynomial g_B contacts the "bottom" boundary. The theorem we wish to prove may now be stated as follows:

Theorem 1: $x_R = \text{maximum}\ (x_T, x_B)$, and further $g_R = g_T = g_B$.

The first part of the theorem, i.e., $x_R = \text{maximum}\ (x_T, x_B)$ may be established as follows. Assume that max $(x_T, x_B) = x_T$. It cannot be the case that $x_R < x_T$ since $R(f, g_T) > x_R$ contrary to hypothesis. Similarly it cannot be that $x_R > x_T$ since in this case either $T(f, g_R) > x_T$ or $B(f, g_R) > x_B$ both contrary to hypothesis. Since it cannot be the case that $x_R > x_T$ or that $x_R > x_T$, it must be that $x_R = x_T$. A similar argument can be advanced for the case max $(x_T, x_B) = x_B$ by simply reversing the roles of x_T and x_B in the above argument.

For the purpose of the following argument let us again assume that $x_R = \max(x_T, x_B) = x_T$. Clearly $g_R = g_T$ and in order to complete the proof of the theorem we must show that $g_B = g_R$. Toward this end we employ a theorem due to Tchebysheff which

says in effect that the quantity $|f(y)-g_R(y)|$ takes on its maximum value at least n + 2 times over the interval $[0,x_R]$, say $x_1<x_2<\ldots<_{n+1}<x_{n+2}=x_R$, and that the sign of the error at successive points alternates. Thus if n is even and $g(x_R)=f(x_R)+\delta$ we have that $g(x_j)=f(x_j)+\delta$ for $j=2i$, $1\leq i\leq\frac{n}{2}+1$ and similarly $g(x_j)=f(x_j)-\delta$ for $j=2i-1$, $1\leq i\leq\frac{n}{2}+1$. We will have our desired result if we demonstrate that $x_{n+1}=x_B$. Let us assume that it is not the case that $x_B=x_{n+1}$ but that $x_{n+1}<x_B<x_R=x_T$. Since x_{n+1} is the rightmost point at which $g_R=f-\delta$ it follows that $g_R(x_B)>g_B(x_B)$. We note that g_R is a nth degree polynomial determined by the n+1 points $(x_1,f(x_1)-\delta)$, $(x_2,f(x_2)+\delta),\ldots,$ $(x_n,f(x_n)+\delta)$, $(x_{n+1},f(x_{n+1})-\delta)$. Similarly g_B is an nth degree polynomial determined by $(x_1,f(x_1)-\delta+\Delta_1)$, $(x_2,f(x_2)+\delta-\Delta_2),\ldots,$ $(x_n,f(x_n)+\delta-\Delta_n)$, $(x_{n+1}, f(x_{n+1})-\delta+\Delta_{n+1})$ in which $\Delta j\geq 0$. Consider the nth degree polynomial h(x) defined by the points $(x_1,\Delta_1),(x_2,-\Delta_2),\ldots,$ $(x_n,-\Delta_n)(x_{n+1},\Delta_{n+1})$. It must be true that $g_B=g_R+h$ since they are all nth degree polynomials and g_B coincides with g_R+h at n+1 points. We note that h(x) changes sign between every pair of points so that all of the roots of h(x) are accounted for and h(x) must be positive for all $x>x_{n+1}$. We have that $h(x_B)>0$ or that $g_B(x_B)>g_R(x_B)$. Thus we have a contradiction resulting from the assumption that $x_B>x_{n+1}$. Since $B(f,g_R)=x_{n+1}$ it cannot be that $x_B<x_{n+1}$ and therefore $x_B=x_{n+1}$ and $g_B=g_R$. Similar arguments can be advanced for the case of n odd and also for the case in which $x_R=x_B$.

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